# MATH 2850: CHAPTER 3: NUMERICAL METHODS

**RECALL:** From algebra: the slope of the line $m$ containing points $(x_0, y_0)$ and $(x_1, y_1)$ is:

$$m = \frac{y_1 - y_0}{x_1 - x_0} \iff m(x_1 - x_0) = y_1 - y_0 \iff y_1 = y_0 + m(x_1 - x_0) \iff y_1 = y_0 + m\,\Delta x$$

That is, $y_1 = \text{initial } y \text{ value} + \text{slope} \cdot \text{change in } x$.

## EULER METHODS:

Euler's Method is a way to **numerically** solve an IVP. The big idea us to 'follow the slope field.'

Consider the IVP: $y' = f(x, y)$, $y(x_0) = y_0$. We start at the point $(x_0, y_0)$.

The slope of the tangent line at $(x_0, y_0)$ is $y' = f(x_0, y_0)$.

We use this slope to predict another point $(x_1, y_1)$ on the solution curve as follows:

We define $x_1 = x_0 + h$ so $h = x_1 - x_0 = \Delta x$.

Next we recall the slope of the tangent line at $(x_0, y_0)$ on the solution curve is, $m = y' = f(x_0, y_0)$. Hence,

$$y_1 = y_0 + m\,\Delta x \iff y_1 = y_0 + f(x_0, y_0)\,h$$

So our new point $(x_1, y_1)$ can be summarized as:

$$x_1 = x_0 + h, \qquad y_1 = y_0 + hf(x_0, y_0)$$

Starting at $(x_1, y_1)$ we can proceed as above to a new point $(x_2, y_2)$ and so on.

Sketch a diagram below!

In general, we get the equations:

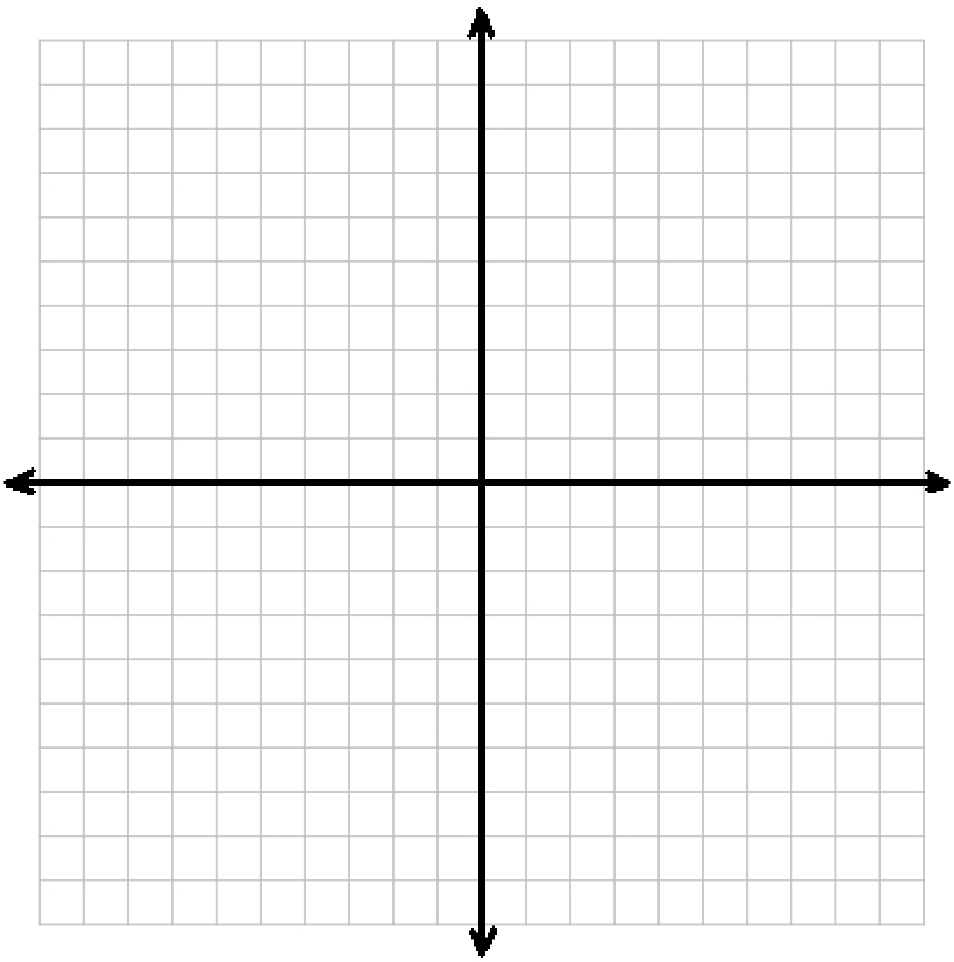
$$x_{i+1} = x_i + h, \qquad y_{i+1} = y_i + hf(x_i, y_i)$$

The value $h$ is called the **step size**. We are starting at $x_0$ and 'stepping' at increments $h$ to some final $x$-value.

**EXAMPLE:** Consider $y' = xy - 1$, $y(0) = 1$. The EUT guarantees a unique solution.

Use Euler's Method to approximate $y(2)$ using $h = 0.5$. Graph your answer below.

| $i$ | $x_i$ | $y_i$ | $f(x_i, y_i)$ | $f(x_i, y_i)h$ | $y_{i+1}$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | | | |
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |



**ERROR BOUND:** The error for Euler's Method is: '$O(h)$' or 'order $h$.'

This means if we halve $h$, we half the error: $\dfrac{h}{2} \implies \dfrac{\text{error}}{2}$

**EXAMPLE:** Use Euler's Method to approximate $y(2)$ using $h = 0.25$ and record your estimate.

**IMPROVED EULER'S (OR HEUN'S) METHOD:**

**IDEA:** Average the slopes at $x_i$ and $x_{i+1}$ to get a better idea of the true location of $y_{i+1}$.

A so-called: 'predictor and corrector' method:

$$x_{i+1} = x_i + h, \qquad y^*_{i+1} = y_i + hf(x_i, y_i), \qquad y_{i+1} = y_i + h\frac{\left[f(x_i, y_i) + f(x_i + h, y^*_{i+1})\right]}{2}$$

Sketch a generic graph of what's happening here!

**EXAMPLE:** Use the Improved Euler's Method to solve $y' = xy - 1$, $y(0) = 1$ with $h = 0.5$.

**ERROR BOUND:** The error for the Improved Euler's Method is: $O(h^2)$.

This means if we halve $h$, we quarter the error: $\frac{h}{2} \implies \frac{\text{error}}{4}$.

**EXAMPLE:** Use the Improved Euler Method to approximate $y(2)$ using $h = 0.25$ and record your estimate.

**RUNGE-KUTTA METHOD:** Starting with the Improved Euler Method:

$$x_{i+1} = x_i + h, \qquad y^*_{i+1} = y_i + hf(x_i, y_i), \qquad y_{i+1} = y_i + h\frac{\left[f(x_i, y_i) + f(x_i + h, y^*_{i+1})\right]}{2}$$

Relabel $k_{i1} = f(x_i, y_i)$, so $y^*_{i+1} = y_i + hk_{i1}$, and $k_{i2} = f(x_i + h, y^*_{i+1}) = f(x_i + h, y_i + hk_{i1})$. Then:

$$y_{i+1} = y_1 + h\left(\frac{k_{i1} + k_{i2}}{2}\right)$$

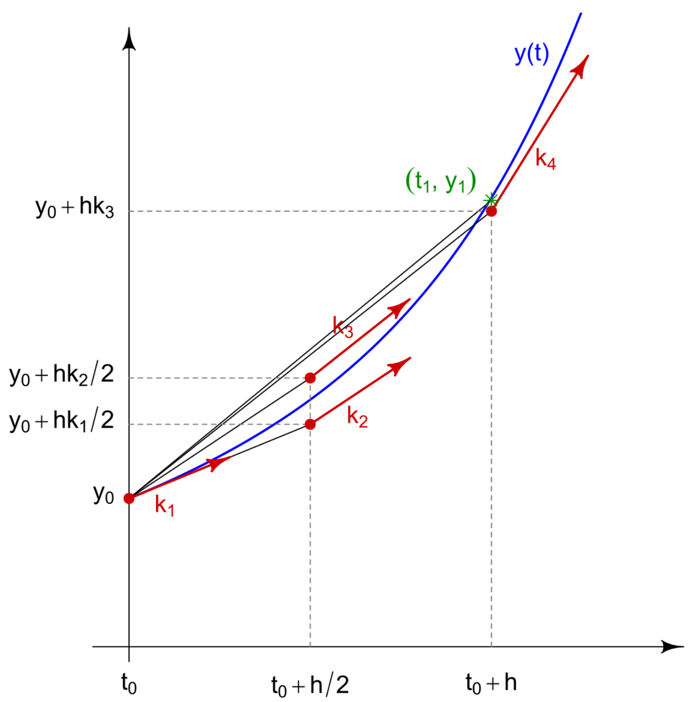
The Runge Kutta Method extends this idea of averaging even further:

$$k_{i1} = f(x_i, y_i), \quad k_{i2} = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{i1}\right), \quad k_{i3} = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{i2}\right), \quad k_{i4} = f(x_i + h, y_i + hk_{i3})$$

$$y_{i+1} = y_i + h\left(\frac{k_{i1} + 2k_{i2} + 2k_{i3} + k_{i4}}{6}\right)$$

Below is a nice visual from HilberTraum[1] to help you visualize the method.

**NOTE:** In the diagram below, the independent variable is $t$, not $x$:




**ERROR BOUND:** The error for the Runge Kutta Method is: $O(h^4)$.

This means if we halve $h$, we divide the error by 16: $\dfrac{h}{2} \implies \dfrac{\text{error}}{16}$.

**EXAMPLE:** Use the Runge Kutta Method to solve $y' = xy - 1$, $y(0) = 1$ with $h = 0.5$ and again with $h = 0.25$.

[1]Own work, CC BY-SA 4.0, see here

**EXAMPLE:** Solve $y' = xy - 1$, $y(0) = 1$. Use a graphing utility to approximate $y(2)$.

How do all of our previous estimates stack up?

Ans: $y = e^{x^2/2}\left(1 - \int_0^x e^{-t^2/2}\,dt\right)$, $y(2) \approx -1.450383\ldots$

**EXAMPLE:** Suppose we wish estimate the value of $\int_a^b f(x)\,dx$.

Defining $y(x) = \int_a^x f(t)\,dt$ we may phrase this problem as finding $y(b)$ for the IVP $y' = f(x)$, $y(a) = 0$.

Revisit Euler's Method, the Improved Euler's Method, and the Runge-Kutta methods in this case.

**NOTE:** Here, $f(x, y) = f(x)$ so the formulas simplify quite nicely.

Revisit your notes from Calculus on 'Numerical Integration'. Do you see any familiar formulas?

**HOMEWORK:** pgs. 106, 117, 124: 7